

Three-quark confinement potential from the Faddeev equation

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In the heavy quark limit of Coulomb gauge QCD and by truncating the Yang-Mills sector to include only dressed two-point functions, an analytic nonperturbative solution to the Faddeev equation for three-quark bound states in the case of equal quark separations is presented. A direct connection between the temporal gluon propagator and the three-quark confinement potential is provided and it is shown that only color singlet qqq (baryon) states are physically allowed.

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I. INTRODUCTION

In order to understand the infrared properties of Quantum Chromodynamics [QCD], the heavy quark sector is a useful area of study. Among the heavy quark correlations, the most basic quantity is the confinement potential between a quark-antiquark pair. At large separations, Wilson loops in lattice calculations exhibit an area law which corresponds to a linearly rising potential whose coefficient, the Wilsonian string tension, can be explicitly related to a hadronic scale [1]. Within continuous functional approaches in Coulomb gauge, recent investigations have shown that (at least under truncation) there is a direct connection between the Green's functions of Yang-Mills theory and the physical string tension that confines quarks [2]. In the Hamiltonian formalism, this relates to both the non-Abelian color Coulomb potential [3], and the temporal Wilson loop [4].

On the other hand, the potential that describes the interaction of three quarks is much less studied than the potential between a quark-antiquark pair. Lattice simulations have indeed shown that the Yang-Mills interaction gives rise to a linearly rising potential between three quarks, but unfortunately a consensus regarding the shape of the gluonic strings connecting the three quarks is still missing: either the strings meet at the so-called Fermat point, which has minimal distances to the three sources (so-called Y configuration) [5–7], or the qqq potential is simply the sum of two-body interactions (so-called Δ configuration) [8]. In the continuum, the only calculations of the three quark potential have been performed at perturbative level, within the so-called potential non-relativistic QCD approach [9]. In Ref. [10], the authors considered the perturbative static potential of three heavy quarks and found that up to NLO, the potential is just the sum of the two-body contributions, whereas at NNLO the three-body contributions do appear, signaling the importance of three-body interactions for understanding the shape of the string in the infrared regime.

The Faddeev equation [11] and its subsequent developments [12, 13] (for an extended review see [14]) provide a general formulation of the relativistic three-body problem. It is a bound state equation (the direct analogue of the homogeneous two-body Bethe-Salpeter equation) and it has been efficiently applied in QCD to study baryon states, via the Green's functions of the theory. Typically, these studies are performed in Landau gauge and, due to the complexity of the equations, they have been mainly restricted to rainbow-ladder truncation, where the kernel is reduced to the single exchange of a dressed gluon. Within this approximation and by employing phenomenological Ansätze for the Yang-Mills part of the theory, the nucleon and Δ properties have been studied [15–18]. Other simplifications include the three-body spectator formalism [14], a Salpeter-type equation with instantaneous interaction [19] or the diquark correlations [20]. Already from the Bethe-Salpeter studies for mesons, it is known that truncating the kernel is not a simple task – the truncation has to be consistent with the symmetry properties of the theory, e.g., the axialvector Ward-Takahashi identity must be satisfied. In contemporary studies, the Bethe-Salpeter kernel has been considered beyond the rainbow-ladder truncation for meson states, including both vertex corrections [21–26] as well as unquenching effects [22, 27–29] and it was found that (apart from meson decay induced by unquenching [28]) the rainbow-ladder approximation works surprisingly well.

For many years, Coulomb gauge studies have been recognized as a promising avenue with which to investigate the nonperturbative regime of QCD [30]. In this gauge, the Gribov-Zwanziger scenario of confinement becomes particularly relevant [31–33]. In this picture, the temporal component of the gluon propagator becomes infrared enhanced, providing for a long range confining force, while the transversal spatial component is infrared suppressed, thus explaining the absence of the asymptotic states in the spectrum. Coulomb gauge is physical, in the sense that in this gauge the system reduces naturally to the physical degrees of freedom (explicitly demonstrated in [33]). Moreover, within the first order functional formalism it has been shown that the total charge of the system is conserved and vanishing, and the well-known energy divergence problem disappears [34]. Within this approach the Dyson-Schwinger equations for the Yang-Mills part of the theory have been derived [35, 36], together with the Slavnov-Taylor identities

[37] and perturbative results have been provided [38]. In addition, the quark sector has been also investigated, within perturbation theory [39] as well as in the heavy mass limit [2]. On the lattice, initial calculations for the Yang-Mills propagators have also become available [43, 44] (see also [45–47]). In particular, the results indicate that the temporal component of the gluon propagator is largely independent of energy (due to noncovariance, in Coulomb gauge the propagators are in general dependent on both the energy and spatial momentum), and it is consistent with a $1/\bar{q}^4$ behavior in the infrared. Moreover, the spatial equal-time gluon propagator is found to be vanishing in the infrared, in agreement with the Gribov’s formula [43, 48]. The lattice calculations support the results obtained from the variational method to the Hamiltonian approach in Yang-Mills theory [3, 49–51].

In this paper, we extend a previous investigation of the heavy quark system in Coulomb gauge [2]. There, the Bethe-Salpeter equation for $\bar{q}q$ bound states was studied with a heavy quark mass expansion (which underlines the Heavy Quark Effective Theory [HQET] [40–42]) at leading order, and a direct connection between the temporal gluon propagator and the string tension was found. Following the same approach, we consider in this work the Faddeev equation for three-quark systems in Coulomb gauge, in the symmetric case (i.e., equal spatial separations between quarks) and with the inclusion of only two-body interactions, at leading order in the mass expansion. We will use the results inspired by the lattice for the Green’s functions of the Yang-Mills sector and in addition, we will employ our previous findings, in particular that nonperturbatively the temporal quark-gluon vertex remains bare under truncation and the kernel of the Bethe-Salpeter equation reduces to the ladder approximation. In this truncated system, we will provide an exact solution to the Faddeev equation, from which the confining potential between three quarks emerges, and we will show that qqq bound states can only exist for $N_c = 3$ colors, i.e. color singlet baryons. (In [2] it was shown that only color-singlet meson and $SU(2)$ qq “baryon” states have finite energy.)

The organization of this paper is as follows. In Sec. II we briefly review relevant results for the heavy quark systems. Starting with the generating functional of Coulomb gauge QCD at leading order in the mass expansion, we review the main steps in the derivation of the heavy quark propagator and the temporal quark-gluon vertex needed in this work. In Sec. III the Faddeev equation for three-quark states is considered. In addition to solving the equation, the pole structure of the quark-baryon vertex is analyzed and an interesting similarity with the $\bar{q}q$ system is discussed. Moreover, a direct connection between the temporal gluon propagator and the physical string tension is found. In Sec. IV a short summary and the concluding remarks will be presented. Some technical details are given in the Appendix.

II. HEAVY QUARK MASS EXPANSION

In this section, let us briefly review some relevant results from [2]. The notations and conventions used in this work are those established in Refs. [2, 35–37, 39]. We work in Minkowski space, with the metric $g_{\mu\nu} = \text{diag}(1, -\vec{1})$. Roman letters (i, j, \dots) refer to spatial indices and superscripts (a, b, \dots) stand for color indices in the adjoint representation of the gauge group. Unless otherwise specified, the Dirac spinor, flavor and (fundamental) color indices are denoted with a common index (α, β, \dots). Configuration space coordinates may be denoted with subscript (x, y, \dots) when no confusion arises. The Dirac γ -matrices satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. The notation γ^i refers to the spatial component, where the minus sign arising from the metric has been explicitly extracted when appropriate.

The explicit quark contribution to the full QCD generating functional within our conventions, can be written [39]

$$\begin{aligned} Z[\bar{\chi}, \chi] = & \int \mathcal{D}\Phi \exp \left\{ i \int d^4x \bar{q}_\alpha(x) \left[i\gamma^0 D_0 + i\vec{\gamma} \cdot \vec{D} - m \right]_{\alpha\beta} q_\beta(x) \right\} \\ & \times \exp \left\{ i \int d^4x [\bar{\chi}_\alpha(x) q_\alpha(x) + \bar{q}_\alpha(x) \chi_\alpha(x)] + i\mathcal{S}_{YM} \right\}. \end{aligned} \quad (2.1)$$

In the above, $\mathcal{D}\Phi$ generically denotes the functional integration measure over all fields and \mathcal{S}_{YM} is the Yang-Mills contribution to the generating functional. q_α denotes the full quark field, \bar{q}_α is the conjugate (or antiquark) field, and $\chi_\alpha, \bar{\chi}_\alpha$ are the corresponding sources. The temporal and spatial components of the covariant derivative (in the fundamental color representation) are given by

$$\begin{aligned} D_0 &= \partial_0 - i g T^a \sigma^a(x), \\ \vec{D} &= \vec{\nabla} + i g T^a \vec{A}^a(x), \end{aligned} \quad (2.2)$$

where \vec{A} and σ refer to the spatial and temporal components of the gluon field, respectively. f^{abc} are the structure constants of the $SU(N_c)$ group, with the Hermitian generators T^a , satisfying $[T^a, T^b] = i f^{abc} T^c$ and normalized via $\text{Tr}(T^a T^b) = \delta^{ab}/2$. For later use we introduce the Casimir factor associated with the quark gap equation:

$$C_F = \frac{N_c^2 - 1}{2N_c}. \quad (2.3)$$

In the following, we briefly sketch the derivation of the quark propagator, in the heavy mass limit. For an extended discussion, the reader is referred to the original work [2]. We start by performing the following decomposition of the quark field

$$q_\alpha(x) = e^{-imx_0} [h(x) + H(x)]_\alpha, \quad h_\alpha(x) = e^{imx_0} [P_+ q(x)]_\alpha, \quad H_\alpha(x) = e^{imx_0} [P_- q(x)]_\alpha \quad (2.4)$$

(similarly for the antiquark field) and introduce the two components h and H , with the help of the spinor projectors

$$P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma^0). \quad (2.5)$$

This corresponds to a particular case of the heavy quark transform underlying HQET with the velocity vector $v^\mu = (1, 0, 0, 0)$ [40], but in the functional approach here it can be regarded simply as an arbitrary decomposition that will turn out to be very useful in Coulomb gauge (precisely, this will lead to the suppression of the spatial gluon propagator in the mass expansion, see below).

We now insert the decomposition, Eq. (2.4), into the generating functional Eq. (2.1), integrate out the H -fields, and make an expansion in the heavy quark mass (throughout this work, we will use the standard term “mass expansion”, instead of “expansion in the *inverse* mass”). At leading order, we get the following expression:

$$\begin{aligned} Z[\bar{\chi}, \chi] = & \int \mathcal{D}\Phi \exp \left\{ i \int d^4x \bar{h}_\alpha(x) [i\partial_{0x} + gT^a \sigma^a(x)]_{\alpha\beta} h_\beta(x) \right\} \\ & \times \exp \left\{ i \int d^4x [e^{-imx_0} \bar{\chi}_\alpha(x) h_\alpha(x) + e^{imx_0} \bar{h}_\alpha(x) \chi_\alpha(x)] + i\mathcal{S}_{YM} \right\} + \mathcal{O}(1/m), \end{aligned} \quad (2.6)$$

where the temporal component of the covariant derivative D_0 has been written explicitly. In the above, we have kept the full quark and antiquark sources (rather than the ones corresponding to the components of the quark field, introduced in HQET). This means that we can use the full gap, Bethe-Salpeter and Faddeev equations of QCD but replace the kernels, propagators and vertices and restrict to the leading order in the mass expansion.

In full QCD (i.e., Coulomb gauge within second order formalism, without the mass expansion and derived from the first order formalism results of Ref. [39]), the quark gap equation is given by [$\bar{d}\omega = d^4\omega/(2\pi)^4$]:

$$\begin{aligned} \Gamma_{\bar{q}q\alpha\delta}^{(0)}(k) = & \Gamma_{\bar{q}q\alpha\delta}^{(0)}(k) + \int \bar{d}\omega \left\{ \Gamma_{\bar{q}q\sigma\alpha\beta}^{(0)a}(k, -\omega, \omega - k) W_{\bar{q}q\beta\gamma}(\omega) \Gamma_{\bar{q}q\sigma\gamma\delta}^b(\omega, -k, k - \omega) W_{\sigma\sigma}^{ab}(k - \omega) \right. \\ & \left. + \Gamma_{\bar{q}qA\alpha\beta i}^{(0)a}(k, -\omega, \omega - k) W_{\bar{q}q\beta\gamma}(\omega) \Gamma_{\bar{q}qA\gamma\delta j}^b(\omega, -k, k - \omega) W_{AAij}^{ab}(k - \omega) \right\} \end{aligned} \quad (2.7)$$

(W_{AA} is the spatial gluon propagator, which will not be regarded here). The gap equation is supplemented by the Slavnov-Taylor identity, which follows from the invariance of the action under a Gauss-BRST transform [2]. In Coulomb gauge, this identity reads:

$$\begin{aligned} k_3^0 \Gamma_{\bar{q}q\sigma\alpha\beta}^d(k_1, k_2, k_3) = & i \frac{k_{3i}}{k_3^2} \Gamma_{\bar{q}qA\alpha\beta i}^a(k_1, k_2, k_3) \Gamma_{\bar{c}c}^{ad}(-k_3) \\ & + \Gamma_{\bar{q}q\alpha\delta}(k_1) \left[\tilde{\Gamma}_{\bar{q};\bar{c}c q}^d(k_1 + q_0, k_3 - q_0; k_2) + i g T^d \right]_{\delta\beta} \\ & + \left[\tilde{\Gamma}_{\bar{q};\bar{c}c \bar{q}}^d(k_2 + q_0, k_3 - q_0; k_1) - i g T^d \right]_{\alpha\delta} \Gamma_{\bar{q}q\delta\beta}(-k_2) \end{aligned} \quad (2.8)$$

where $k_1 + k_2 + k_3 = 0$, q_0 is an arbitrary energy injection scale (arising from the noncovariance of Coulomb gauge [37]), $\Gamma_{\bar{c}c}$ is the ghost proper two-point function, $\tilde{\Gamma}_{\bar{q};\bar{c}c q}$ and $\tilde{\Gamma}_{\bar{q};\bar{c}c \bar{q}}$ are ghost-quark kernels associated with the Gauss-BRST transform.

Now, as a consequence of the Coulomb gauge decomposition, Eq. (2.4), the part of the generating functional given by Eq. (2.6) corresponding to the tree-level spatial quark gluon vertex $\Gamma_{\bar{q}qA}^{(0)}$ is contained within the $\mathcal{O}(1/m)$ contribution which is here neglected. Under the further assumption that the pure Yang-Mills vertices may be neglected, the Dyson-Schwinger equation for the nonperturbative spatial quark-gluon vertex then furnishes the result that $\Gamma_{\bar{q}qA} \sim \mathcal{O}(1/m)$ (see [2] for a complete discussion and justification of this truncation). Similarly, the ghost-quark kernels can be neglected. Thus, under our truncation scheme, the Slavnov-Taylor identity reduces to

$$k_3^0 \Gamma_{\bar{q}q\sigma\alpha\beta}^d(k_1, k_2, k_3) = \Gamma_{\bar{q}q\alpha\delta}(k_1) [i g T^d]_{\delta\beta} - [i g T^d]_{\alpha\delta} \Gamma_{\bar{q}q\delta\beta}(-k_2) + \mathcal{O}(1/m). \quad (2.9)$$

This is then inserted into Eq. (2.7), together with the tree-level quark proper two-point function

$$\Gamma_{\bar{q}q\alpha\beta}^{(0)}(k) = i\delta_{\alpha\beta} [k_0 - m] + \mathcal{O}(1/m) \quad (2.10)$$

and the tree level quark gluon vertex

$$\Gamma_{\bar{q}q\sigma\alpha\beta}^{(0)a}(k_1, k_2, k_3) = [gT^a]_{\alpha\beta} + \mathcal{O}(1/m) \quad (2.11)$$

that follow from the generating functional Eq. (2.6). The general form of the nonperturbative temporal gluon propagator is given by [36]:

$$W_{\sigma\sigma}^{ab}(\vec{k}) = \delta^{ab} \frac{i}{\vec{k}^2} D_{\sigma\sigma}(\vec{k}^2). \quad (2.12)$$

Lattice results, and also more formal consideration in continuum show that the dressing function $D_{\sigma\sigma}$ has some part that is independent of energy [52] and moreover, $D_{\sigma\sigma}$ is infrared divergent and likely to behave as $1/\vec{k}^2$ for vanishing \vec{k}^2 (the explicit form of $D_{\sigma\sigma}$ will only be needed in the last step of the calculation). Putting all this together, we find the following solution to Eq. (2.7) for the heavy quark propagator:

$$W_{\bar{q}q\alpha\beta}(k_0) = \frac{-i\delta_{\alpha\beta}}{[k_0 - m - \mathcal{I}_r + i\varepsilon]} + \mathcal{O}(1/m), \quad (2.13)$$

with the (implicitly regularized, denoted by “ r ”) constant [$d\vec{\omega} = d^3\vec{\omega}/(2\pi)^3$]

$$\mathcal{I}_r = \frac{1}{2}g^2C_F \int_r \frac{d\vec{\omega} D_{\sigma\sigma}(\vec{\omega})}{\vec{\omega}^2} + \mathcal{O}(1/m). \quad (2.14)$$

When solving Eq. (2.7), the ordering of the integration is set such that the temporal integral is performed first, under the condition that the spatial integral is regularized and finite. Inserting the solution Eq. (2.13) into the Slavnov-Taylor identity, we find that the temporal quark-gluon vertex remains nonperturbatively bare:

$$\Gamma_{\bar{q}q\sigma\alpha\beta}^a(k_1, k_2, k_3) = [gT^a]_{\alpha\beta} + \mathcal{O}(1/m). \quad (2.15)$$

The propagator Eq. (2.13) has a couple of striking features, which have been emphasized in [2]. Firstly, due to the mass expansion, we only have a single pole in the complex k_0 -plane, as opposed to the conventional quark propagator, which possesses a pair of simple poles. Hence, it is necessary to explicitly define the Feynman prescription. From Eq. (2.13) it then follows that the closed quark loops (virtual quark-antiquark pairs) vanish due to the energy integration, which implies that the theory is quenched in the heavy mass limit:

$$\int \frac{dk_0}{[k_0 - m - \mathcal{I}_r + i\varepsilon][k_0 + p_0 - m - \mathcal{I}_r + i\varepsilon]} = 0. \quad (2.16)$$

Secondly, the propagator Eq. (2.13) is diagonal in the outer product of the fundamental color, flavor and spinor spaces – physically this corresponds to the decoupling of the spin from the heavy quark system. In fact, $W_{\bar{q}q}^{(0)}$ is identical to the heavy quark tree-level propagator [40] up to the appearance of the mass term, and this is due to the fact that in HQET one uses the sources for the large h -fields directly, while we retain the sources of the full quark fields. Finally, let us emphasize that the position of the pole has no physical meaning since the quark can never be on-shell. The poles in the quark propagator are situated at infinity (thanks to \mathcal{I}_r as the regularization is removed) meaning that either one requires infinite energy to create a quark from the vacuum or, if a hadronic system is considered, only the relative energy is important. Indeed, it was shown some time ago [53] that the divergence of the absolute energy has no physical meaning and only the relative energy (derived from the Bethe-Salpeter equation) must be considered. It is precisely the cancellation of these divergent constants that distinguishes between physical and unphysical poles.

We also note that for the antiquark propagator the opposite Feynman prescription is assigned such that the Bethe-Salpeter equation for the quark-antiquark states has a physical interpretation of a bound state equation. There, the quark and the antiquark do not create a virtual quark-antiquark pair, but a system composed of two separate unphysical particles (in the sense that they are not connected by a primitive vertex). Moreover, the Bethe-Salpeter kernel reduces to the ladder truncation [2]. The reason is the cancellation of the so-called crossed box contributions (i.e., nonplanar diagrams that contain any combinations of nontrivial interactions allowed within our truncation scheme) due to the temporal integration performed over multiple propagators with the same relative sign for the Feynman prescription (similar to Eq. (2.16), but in this case the terms originate from internal quark or antiquark propagators).

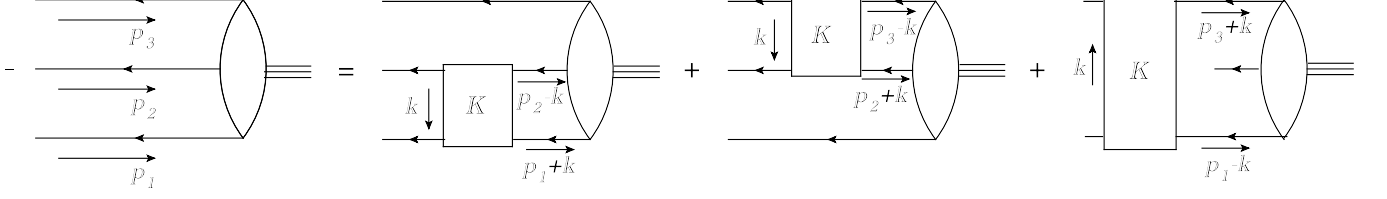


FIG. 1: Faddeev equation for three quark bound states. Solid lines represent the quark propagator, the box represents the diquark kernel K and the ellipse represents the Faddeev vertex function with the bound state leg depicted by a triple-line. See text for details.

III. FADDEEV EQUATIONS FOR THREE-QUARK STATES

Let us now consider the Faddeev equation for three-quark bound states. In this work, we employ only the permuted two quark kernels K (which coincide with kernel appearing in the Bethe-Salpeter equation for diquark states) and neglect the three-quark irreducible diagrams, i.e., genuine three-body forces. This approximation is also motivated by the fact that in the quark-diquark model the binding energy is assumed to be mainly provided by the two-quark correlations [54]. In this truncation, the Faddeev equation reads (see also Fig. 1):

$$\begin{aligned} \Gamma_{\alpha\beta\gamma}(p_1, p_2, p_3; P) = & - \int \bar{d}k \left\{ K_{\beta\alpha; \alpha' \beta'}(k) W_{\bar{q}q \alpha' \alpha''}(p_1 + k) W_{\bar{q}q \beta' \beta''}(p_2 - k) \Gamma_{\alpha'' \beta'' \gamma}(p_1 + k, p_2 - k, p_3; P) \right. \\ & + K_{\gamma\beta; \beta' \gamma'}(k) W_{\bar{q}q \beta' \beta''}(p_2 + k) W_{\bar{q}q \gamma' \gamma''}(p_3 - k) \Gamma_{\alpha\beta'' \gamma''}(p_1, p_2 + k, p_3 - k; P) \\ & \left. + K_{\alpha\gamma; \gamma' \alpha'}(k) W_{\bar{q}q \gamma' \gamma''}(p_3 + k) W_{\bar{q}q \alpha' \alpha''}(p_1 - k) \Gamma_{\alpha\beta'' \gamma''}(p_1 - k, p_2, p_3 + k; P) \right\} \end{aligned} \quad (3.1)$$

where p_1, p_2, p_3 are the momenta of the quarks, $P = p_1 + p_2 + p_3$ is the pole 4-momentum of the bound baryon state and Γ is the so-called quark-baryon Faddeev vertex for the particular bound state under consideration and whose indices denote explicitly only its quark content. Due to the fact that in the heavy mass limit the spin degrees of freedom decouple from the system, at leading order in the mass expansion the Faddeev baryon amplitude $\Gamma_{\alpha\beta\gamma}$ becomes a Dirac scalar, similar to the heavy quark propagator Eq. (2.13). The explicit momentum dependence of the kernels K is abbreviated for notational convenience. As in the homogeneous Bethe-Salpeter equation, the integral equation depends only parametrically on the total four momentum P .

As discussed at the end of the previous Section, the kernel K reduces to the ladder approximation (constructed via gluon exchange) and it reads

$$K_{\beta\alpha; \alpha' \beta'}(k) = \Gamma_{\bar{q}q \sigma \alpha \alpha'}^a W_{\sigma\sigma}^{ab}(\vec{k}) \Gamma_{\bar{q}q \sigma \beta \beta'}^b = g^2 T_{\alpha\alpha'}^a W_{\sigma\sigma}^{ab}(\vec{k}) T_{\beta\beta'}^b \quad (3.2)$$

with the temporal gluon propagator and the temporal quark-gluon vertex given by Eq. (2.12) and Eq. (2.15), respectively. Similar to the Bethe-Salpeter equation for meson bound states, the energy independence of this propagator will turn to be crucial in the derivation of the confining potential.

Let us now have a closer look at the energy dependence of the equation Eq. (3.1). In the meson case, since the Bethe-Salpeter kernel was energy independent, it was straightforward to show that the Bethe-Salpeter vertex itself did not contain an energy dependent part. This observation was then used to calculate the confining potential from the $\bar{q}q$ Bethe-Salpeter equation, via a simple analytical integration over the relative energy variable. Unfortunately this approach cannot be extended to baryon states: despite the instantaneous kernel, a relative energy dependence still remains and thus one cannot assume an energy-independent Faddeev vertex. Therefore, in order to proceed, we make the following separable Ansatz for the Faddeev vertex:

$$\Gamma_{\alpha\beta\gamma}(p_1, p_2, p_3; P) = \Psi_{\alpha\beta\gamma} \Gamma_t(p_1^0, p_2^0, p_3^0; P) \Gamma_s(\vec{p}_1, \vec{p}_2, \vec{p}_3; P) \quad (3.3)$$

where we have introduced a purely antisymmetric (in the quark legs) color factor Ψ (the possible baryon color index is omitted) and the symmetric (Dirac scalar) temporal and spatial components Γ_t and Γ_s , respectively.

Inserting the explicit form of the kernel Eq. (3.2) and the quark-baryon vertex Ansatz Eq. (3.3), the Faddeev equation Eq. (3.1) can be explicitly written as (for simplicity we drop the label P in the arguments of the vertex functions):

$$\begin{aligned} \Gamma_{\alpha\beta\gamma}(p_1, p_2, p_3) = & -g^2 T_{\alpha\tau}^a T_{\beta\kappa}^a \Psi_{\tau\kappa\gamma} \int \bar{d}k W_{\sigma\sigma}(\vec{k}) W_{\bar{q}q}(p_1 + k) W_{\bar{q}q}(p_2 - k) \Gamma_t(p_1^0 + k_0, p_2^0 - k_0, p_3^0) \Gamma_s(\vec{p}_1 + \vec{k}, \vec{p}_2 - \vec{k}, \vec{p}_3) \\ & + \text{cyclic permutations,} \end{aligned} \quad (3.4)$$

where the explicit color structure has been extracted ($W_{\sigma\sigma}^{ab} = \delta^{ab}W_{\sigma\sigma}$, $W_{\bar{q}q\alpha\beta} = \delta_{\alpha\beta}W_{\bar{q}q}$).

With the help of the Fierz identity for the generators T^a

$$2 [T^a]_{\alpha\beta} [T^a]_{\delta\gamma} = \delta_{\alpha\gamma}\delta_{\delta\beta} - \frac{1}{N_c}\delta_{\alpha\beta}\delta_{\delta\gamma}, \quad (3.5)$$

the color structure can be written as

$$T_{\alpha\alpha'}^a T_{\beta\beta'}^a \Psi_{\alpha'\beta'\gamma} = -C_B \Psi_{\alpha\beta\gamma} \quad (3.6)$$

with $C_B = (N_c + 1)/2N_c$, where N_c is the number of colors, yet to be identified (i.e., the baryon is not assumed to be a color singlet).

In the next step we perform the Fourier transform for the spatial part of the equation, recalling that the heavy quark propagator is only a function of energy. We define the coordinate space vertex function via its Fourier transform

$$\Gamma_s(\vec{p}_1, \vec{p}_2, \vec{p}_3) = \int d\vec{x}_1 d\vec{x}_2 d\vec{x}_3 e^{-i\vec{p}_1 \cdot \vec{x}_1 - i\vec{p}_2 \cdot \vec{x}_2 - i\vec{p}_3 \cdot \vec{x}_3} \Gamma_s(\vec{x}_1, \vec{x}_2, \vec{x}_3) \quad (3.7)$$

(similarly for $W_{\sigma\sigma}$, as in [2]) such that

$$\int d\vec{k} W_{\sigma\sigma}(\vec{k}) \Gamma_s(\vec{p}_1 + \vec{k}, \vec{p}_2 - \vec{k}, \vec{p}_3) = \int d\vec{x}_1 d\vec{x}_2 d\vec{x}_3 e^{-i\vec{p}_1 \cdot \vec{x}_1 - i\vec{p}_2 \cdot \vec{x}_2 - i\vec{p}_3 \cdot \vec{x}_3} W_{\sigma\sigma}(\vec{x}_2 - \vec{x}_1) \Gamma_s(\vec{x}_1, \vec{x}_2, \vec{x}_3). \quad (3.8)$$

Clearly, the component Γ_s trivially simplifies (as before, we have separated the temporal and spatial integrals, under the assumption the spatial integral is regularized and finite) and the equation Eq. (3.4) reduces to $[\vec{d}k_0 = dk_0/(2\pi)]$:

$$\Gamma_t(p_1^0, p_2^0, p_3^0) = g^2 C_B W_{\sigma\sigma}(\vec{x}_2 - \vec{x}_1) \int d\vec{k}_0 W_{\bar{q}q}(p_1^0 + k_0) W_{\bar{q}q}(p_2^0 - k_0) \Gamma_t(p_1^0 + k_0, p_2^0 - k_0, p_3^0) + \text{cyclic permutations}. \quad (3.9)$$

At this point we make a further simplification, motivated by the symmetry of the three-quark system: we restrict to a particular geometry, namely to equal quark separations, i.e. $|\vec{r}| = |\vec{x}_2 - \vec{x}_1| = |\vec{x}_3 - \vec{x}_2| = |\vec{x}_1 - \vec{x}_3|$. By inserting the explicit form of the quark propagators, Eq. (2.13), we have

$$\Gamma_t(p_1^0, p_2^0, p_3^0) = -g^2 C_B W_{\sigma\sigma}(|\vec{r}|) \int d\vec{k}_0 \frac{\Gamma_t(p_1^0 + k_0, p_2^0 - k_0, p_3^0)}{[p_1^0 + k_0 - m - \mathcal{I}_r + i\varepsilon][p_2^0 - k_0 - m - \mathcal{I}_r + i\varepsilon]} + \text{cyclic permutations}. \quad (3.10)$$

Assuming that the vertex Γ_t is symmetric under permutation of quark legs, an Ansatz that satisfies this equation is:

$$\Gamma_t(p_1^0, p_2^0, p_3^0) = \sum_{i=1,2,3} \frac{1}{2P_0 - 3(p_i^0 + m + \mathcal{I}_r) + i\varepsilon}. \quad (3.11)$$

Since the explicit derivation is rather technical, we only give here the solution and defer the details to the Appendix.

Notice that in the expression Eq. (3.11) there are simple poles (in the energy) present. These poles however do not occur for finite energies and cannot be physical. As discussed, this is also the case for the quark propagator. Intuitively, when a single heavy quark is pulled apart from the system, the qqq state becomes equivalent (i.e., it has the same color quantum numbers) to the $\bar{q}q$ system in the sense that the remaining two quarks form a diquark which for $N_c = 3$ would be a color antitriplet configuration, and hence the physical interpretation of the vertex Eq. (3.11) can be directly related to the heavy quark propagator Eq. (2.13): the presence of the single pole in Eq. (3.11) simply means that this cannot have the meaning of physical propagation (this would require a covariant double pole). Moreover, the divergent constant \mathcal{I}_r appearing in the absolute energy does not contradict the physics – the only relevant quantity is the relative energy of the three quark system.

With this Ansatz at hand, we return to the formula Eq. (3.10), insert the definitions Eq. (2.12) and Eq. (2.14) for $W_{\sigma\sigma}(\vec{x})$ and \mathcal{I}_r , and arrive at the following solution for the bound state energy P_0 , in the case of equal quark separations:

$$P_0 = 3m + \frac{3}{2}g^2 \int d\vec{\omega} \frac{D_{\sigma\sigma}(\vec{\omega})}{\vec{\omega}^2} [C_F - 2C_B e^{i\vec{\omega} \cdot \vec{r}}]. \quad (3.12)$$

Since the quarks cannot be prepared as isolated states, the only possibilities for the qqq state are either that the system is confined (i.e., the bound state energy P_0 increases with the separation), or the system is physically not allowed (i.e., the energy P_0 is infinite). From the formula Eq. (3.12) and knowing that $D_{\sigma\sigma}(\vec{\omega})$ is infrared enhanced, it is clear that in order to have an infrared confining solution (corresponding to a convergent three-momentum integral), the condition

$$C_B = \frac{C_F}{2} \quad (3.13)$$

must be satisfied. This is fulfilled for $N_c = 3$ colors, implying that $\Psi_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma}$ and that the baryon is a color singlet (confined) bound state of three quarks; otherwise, for $N_c \neq 3$ the energy of the the system is infinite for any separation $|\vec{r}|$.

Assuming that in the infrared $D_{\sigma\sigma}(\vec{\omega}) = X/\vec{\omega}^2$ (as indicated by the lattice data [43–47] and by the variational calculations in the continuum [3]), where X is some combination of constants, it is straightforward to perform the integration on the right hand side of Eq. (3.12), with the result that for large separation $|\vec{r}|$:

$$P_0 = 3m + \frac{3}{2} \frac{g^2 C_F X}{8\pi} |\vec{r}|. \quad (3.14)$$

This mimics the previous findings for $\bar{q}q$ systems, namely that there exists a direct connection between the string tension and the nonperturbative Yang-Mills Green's functions (at least under truncation). In this case, the standard term “string tension” refers to the coefficient of the three-body linear confinement term $\sigma_{3q}|\vec{r}|$. Also, comparing with the result of Ref. [2], we find that the string tension corresponding to the qqq system is 3/2 times that of the $\bar{q}q$. To our knowledge, no direct comparison between the string tensions of the two systems has been made and hence this relation would be interesting to investigate on the lattice. The appearance of three times the quark mass stems from the presence of the mass term in the heavy quark propagator Eq. (2.13) which enters the Faddeev equation, and this originates from the fact that in the generating functional Eq. (2.6) we have retained the full source terms (in contrast with the HQET, where one uses sources for the h -fields directly).

IV. SUMMARY AND CONCLUSIONS

In this paper, the Faddeev equation, truncated to include only two-body interactions for three-quark states in a symmetric configuration, has been considered. At leading order in the heavy quark mass expansion of Coulomb gauge QCD and with the truncation to include only the nonperturbative two-point functions of the Yang-Mills sector (and neglect all the pure Yang-Mills vertices and higher order functions) the three-quark confining potential has been derived and a direct connection between the temporal gluon propagator and the physical string tension has been provided. It was found that, as in the case of $\bar{q}q$ systems, the bound state energy increases linearly with the distance for large separations, and that the coefficient was 3/2 times that of the $\bar{q}q$ system.

Due to the absence of the three-body interactions and the restriction to a symmetric configuration, no statement about the shape of the confining string (Δ or Y configuration) can be made. From this point of view, a very interesting extension of this work would be to consider general separations between quarks and to explicitly include the Yang-Mills vertices in the combined system of Dyson-Schwinger and Slavnov-Taylor identities, together with the Faddeev equations, and see if one can extract any information about the shape of the string. Also, by the inclusion of the Yang-Mills vertices one can investigate the charge screening mechanism, i.e. study how the value of the string tension modifies.

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Appendix: Temporal component of the quark-baryon vertex

In this appendix we present the explicit derivation of the energy-dependent part of the Faddeev vertex, Eq. (3.11). We start with Eq. (3.10) and consider the first of the permutations of the energy integral:

$$I = - \int d k_0 \frac{1}{[p_1^0 + k_0 - m - \mathcal{I}_r + i\varepsilon][p_2^0 - k_0 - m - \mathcal{I}_r + i\varepsilon]} \Gamma_t(p_1^0 + k_0, p_2^0 - k_0, p_3^0). \quad (\text{A.1})$$

Using

$$\frac{1}{[z + a + i\varepsilon][z + b + i\varepsilon]} = \frac{1}{(b - a)} \left\{ \frac{1}{z + a + i\varepsilon} - \frac{1}{z + b + i\varepsilon} \right\} \quad (\text{A.2})$$

and shifting the integration variables, we find that the integral I , Eq. (A.1), depends only on the momentum p_3^0 (and implicitly on the bound state energy of the system P_0). Explicitly, it reads (using the symmetry of Γ_t):

$$I = - \frac{2}{[P_0 - p_3^0 - 2(m + \mathcal{I}_r)]} \int d k_0 \frac{1}{[k_0 + P_0 - p_3^0 - m - \mathcal{I}_r + i\varepsilon]} \Gamma_t(P_0 - p_3^0 + k_0, -k_0, p_3^0). \quad (\text{A.3})$$

Replacing this in the equation Eq. (3.10), we find:

$$\Gamma_t(p_1^0, p_2^0, p_3^0) = -2g^2 C_B W_{\sigma\sigma}(|\vec{r}|) \sum_{i=1,2,3} \frac{1}{[P_0 - p_i^0 - 2(m + \mathcal{I}_r)]} \int d k_0 \frac{\Gamma_t(P_0 - p_i^0 + k_0, -k_0, p_i^0)}{[P_0 - p_i^0 + k_0 - m - \mathcal{I}_r + i\varepsilon]}. \quad (\text{A.4})$$

The form of the equation Eq. (A.4) suggests that the function Γ_t can be expressed as a symmetric sum

$$\Gamma_t(p_1^0, p_2^0, p_3^0) = f(p_1^0) + f(p_2^0) + f(p_3^0), \quad (\text{A.5})$$

such that the integral equation for Γ_t (function of three variables) is reduced to an integral equation for the function f (of only one variable). The function $f(p_i^0)$ should be chosen such that the integral on the right hand side of the equation Eq. (A.4) generates a factor proportional to $[P_0 - p_i^0 - 2(m + \mathcal{I}_r)]$, to cancel the corresponding factor in the denominator. To examine this possibility, we impose the following condition:

$$\frac{1}{[P_0 - p_i^0 - 2(m + \mathcal{I}_r)]} \int d k_0 \frac{f(P_0 - p_i^0 + k_0) + f(-k_0) + f(p_i^0)}{[P_0 - p_i^0 + k_0 - m - \mathcal{I}_r + i\varepsilon]} = - \frac{\alpha i}{P_0 - 3(m + \mathcal{I}_r)} f(p_i^0) \quad (\text{A.6})$$

where α is a (dimensionless) positive constant which remains to be determined. Rearranging the terms to factorize the function $f(k_0)$, the above equation can be rewritten as

$$\int d k_0 f(k_0) \left[\frac{1}{k_0 - m - \mathcal{I}_r + i\varepsilon} + \frac{1}{P_0 - p_i^0 - k_0 - m - \mathcal{I}_r + i\varepsilon} \right] = (-i) \frac{(2\alpha - 1)P_0 - 2\alpha p_i^0 + (3 - 4\alpha)(m + \mathcal{I}_r)}{2[P_0 - 3(m + \mathcal{I}_r)]} f(p_i^0). \quad (\text{A.7})$$

Then the most obvious Ansatz for the function f is

$$f(k_0) = \frac{1}{(2\alpha - 1)P_0 - 2\alpha k_0 + (3 - 4\alpha)(m + \mathcal{I}_r) + i\varepsilon} \quad (\text{A.8})$$

such that on the right hand side of the equation Eq. (A.7) the numerator is cancelled by $f(p_i)$. The next step is to complete the integration on the left hand side, which gives (note that the ε prescription is chosen such that only the first term in the bracket survives – the integration must not give rise to any new terms containing the energy p_i^0):

$$\int d k_0 f(k_0) \frac{1}{k_0 - m - \mathcal{I}_r + i\varepsilon} = - \frac{i}{(2\alpha - 1)P_0 + (3 - 6\alpha)(m + \mathcal{I}_r)}. \quad (\text{A.9})$$

It is then straightforward to compare Eq. (A.7) and Eq. (A.9) and find that the equality is satisfied for $\alpha = 3/2$, leading to the expression for the vertex Γ_t used in the text.

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